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APOLLO NAVIGATIONAL ACCURACY IN LUNAR ORBIT INCLUDING LANDMARK UPDATING

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SUMMARY

The analytic description of an orbital navigation program that will simulate the estimation of position and velocity of an Apollo spacecraft while in orbit about the moon or the earth is presented. The inertial positions of a series of landmarks are observed and used as the navigational information, which is processed by linear filter theory techniques. The landmark positions are also estimated and their uncertainties in location are reduced.

The program is applied in this report to the problem of estimating the position and velocity of an Apollo spacecraft while in orbit about the moon. Using realistic initial values for the uncertainties in landmark positions, it is shown for a typical case that the RMS position and velocity estimation errors at Lunar Excursion Module separation are about three times as high as the unrealistic, overoptimistic case of assuming perfectly known landmarks. The landmark estimation uncertainties are decreased by about two-thirds of their initial uncertainties.

INTRODUCTION

The uncertainty in the locations of lunar landmarks will affect the accuracy of spacecraft navigation while in lunar orbit. The effect of these uncertainties can be reduced by including the landmark vector into the system state vector. The resulting nine-component state vector includes the three components of spacecraft position, three of spacecraft velocity, and three components of landmark position. Linear filter theory (see Appendix A) is then used to obtain the best estimate of spacecraft position and velocity along with the best estimate of the landmark position. As the estimate of the landmark position improves, it approaches the actual position with corresponding improvement in spacecraft position and velocity estimates. The amount of improvement is limited by the accuracy of the optical instrument used in measuring the orientation of the landmark with respect to an inertial system with origin at the spacecraft.

Such a scheme for navigation in the vicinity of the moon has two very useful functions:

(1) The navigation function itself, which is of course, the immediate problem.

(2) The mapping function, which enables lunar landmarks to be determined to a higher degree of accuracy. This means, of course, that on a subsequent mission, a set of landmarks that are known to greater precision will exist.

The purpose of this note is to present an analytic description of a spacecraft orbital navigation program and a determination of the current estimate of Apollo lunar orbit navigational accuracy.

There are several error sources that are involved in lunar orbital navigation which should be considered. Numerical values assumed for these errors are listed in Table I along with the appropriate reference. These may be divided into three groups:

(1) The landmark location errors

(2) Navigator acquisition errors

(3) The navigation system errors.

The landmark location errors may also be broken down into convenient classifications. The first source of error is the uncertainty in the location of the landmarks relative to the moon-fixed system. The second source is due to the uncertainty in the inertial location of the moon (earth-moon distance, the orientation of the moon on its spin-axis, and the uncertainty in the spin rate of the moon.

The navigation system errors are due to the inaccuracy of the optical instrument even under ideal conditions, and to the mis-orientation of the inertial platform. For purposes of this study both of these errors have been lumped into the standard deviation of the instrument.

The navigator acquisition errors should be considered alone, although for simulation purposes these errors will be included only in the landmark location errors. This follows to a large extent from the fact that these errors will likely affect the results most severely by not acquiring the landmark properly. These errors then show up as errors in landmark location.

There are, of course, other sources of navigator errors to which it is difficult to assign numbers. For example, eye fatique, lack of sleep, and general mis-orientation of the astronaut due to his new environment; all would have effects on the actual navigation. However, these error sources are not considered in the present investigation.

OPERATIONAL CONSTRAINTS

The operational constrains that must be considered in the simulation are visibility and sighting frequency. The only visibility constraint that will be imposed is that the landmark must be sunlit. The closer the moon is to being full, the larger is the number of landmarks available for navigation. Obviously, this constraint reflects itself back to launch date and flight time.

The spacecraft is assumed to be in an 80-n.mi. circular orbit. At this altitude the distance from a spacecraft to a landmark on the horizon is about 23 degrees [see sketch 1(a)]. In practice, lunar landmarks will probably be difficult to identify and acquire until the spacecraft is much closer to the landmark than 23 degrees. It will be assumed that no observation will be made unless the landmark is within a five-degree cone of the spacecraft, as shown in sketch 1(a). For a five-degree cone it is assumed that a maximum of about four sightings can be made if the sightings are approximately one minute apart.

THE SYSTEM EQUATIONS

The system configuration is the position and velocity vectors of the spacecraft (trajectory state) plus the position vector of a single landmark while the spacecraft is actually in a position to observe the landmark (as defined in the preceding section). When the landmark is not visible the system will refer to the trajectory state only.

It is assumed that a statistical correlation exists between the errors in the trajectory state and errors in the landmark position. However, when a landmark is no longer being observed, it is assumed that this correlation vanishes because of subsequent measurements on other landmarks. It is also assumed that the estimated state is close enough to the actual state so that the two states may be considered to be linearly related. The deviations of the measured quantities from their actual values actual values are linearly related to the deviation of the estimated system state from its actual value (Appendix B). With these last two assumptions the system may be estimated by means of the optimal linear filter theory, (reference 2). It is further assumed that the correlation matrix of the errors in a lanmark's position is used to initiate the computation when the same landmark is next observed.

The equations of motion for the actual state are:

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x},t), \text{ with the initial conditions } \underline{x}(t_o) + \underline{\epsilon}(t_o) \qquad (1)$$

and for the estimated state

$$\dot{\hat{\underline{x}}}(t) = \underline{f}(\hat{x}, t), \text{ with the initial conditions } \hat{\underline{x}}(t_o) = \underline{x}(t_o) \qquad (2)$$

where $\underline{x}(t_o)$ is specified and $\underline{\epsilon}(t_o)$ is a vector of random position and velocity deviations. In reality, $\underline{\epsilon}$ is impossible to determine. For purposes of simulation $\underline{\epsilon}$ may be computed from the eigen-values and eigenvectors of the initial covariance matrix $E(t_o)$ and a set of normally distributed random numbers as shown in reference 3.

The vector $\underline{x}$ is called the state vector and may be expressed as

$$\underline{x}^T = (\underline{r}, \underline{v}, \underline{x}_L) \tag{3a}$$

when a landmark is being observed; or as

$$\underline{x}^T = (\underline{r}, \underline{v}) \tag{3b}$$

when no observations are being taken.

The covariance matrix of the errors in the estimate of the state may be propagated by the matrix differential equation,

$$E(t) = F(t)\, E(t) + E(t)\, F^T(t),\ E(t_o) \text{ specified} \tag{4}$$

The covariance matrix is defined by,

$$E = \mathcal{E}\,(\underline{e}\,\underline{e}^T) \tag{5}$$

where

$$\underline{e} = \hat{\underline{x}} - \underline{x} \tag{6}$$

From this definition it is seen that E is a symmetric matrix, which for this problem is 9x9 when a landmark is visible, 6x6 when no landmark is visible. For convenience E may be broken down into augmented matrices,

$$E = \begin{bmatrix} E_V & E_{VL} \\ E^T_{VL} & E_L \end{bmatrix} \tag{7}$$

E_V is the covariance matrix of the trajectory and is considered at all times. E_L is the covariance matrix of the landmark being observed, and E_{VL} is the matrix which represents the correlation between the errors in the estimate of the trajectory and those of the landmark.

The matrix F is composed of the partial derivatives of $\underline{f}$ with respect to the state vector $\underline{x}$.

When no measurements are being made the state vector is updated by integration of equation (2) and the covariance matrix by equation (4). Between measurements on the same landmark on the same orbit, equation (4) is integrated in its entirety; between measurements of different landmarks only E_V is integrated.

The optimal linear filter (Appendix A) is employed at a measurement time. The state is updated by equation (A8) and the covariance matrix by equation (A9). The initial conditions for (2) and (4) are then reset by these updated values and the integration of (2) and (4) proceeds until the next measurement.

Although the program to be described refers to using lunar landmarks as navigational information, the analysis is equally valid for determining the orbit of a spacecraft about the earth. The coordinate system in which the state variables are described is as follows: the position and velocity of the spacecraft ($\underline{r}$ and $\underline{v}$) are in a selenocentric Cartesian coordinate system. The landmark vector $\underline{x}_L$ is in a selenographic (moon-fixed) polar coordinate system. When $\underline{x}_L$ is expressed in its selenographic Cartesian components ($\underline{x}_L'$) it may be operated on by a transformation matrix A, that will express it in the selenocentric system, yielding $\underline{r}_L = A\underline{x}_L'$.

By expressing $\underline{r}_L$ in the selenocentric system,

$$\dot{\underline{x}}_L = 0 \tag{8}$$

so that f_7, f_8, and f_9 of the matrix F in equation (4) are zero. f_1, f_2, and f_3 are just the components of $\underline{v}$; and f_4, f_5, and f_6 are just the gravitational accelerations due to the primary attracting body (the moon in this case) plus the perturbations due to other bodies and their nonhomogeneities.

THE LINEARIZATION OF THE SYSTEM EQUATIONS

Let the actual trajectory be considered a nominal about which the equations of motion will be linearized. Taking first order deviations of equation (2),

$$\delta \dot{\hat{\underline{x}}} = F(t)\, \delta \hat{\underline{x}}(t) \qquad (9)$$

where

$$\delta \hat{\underline{x}}(t) = \hat{\underline{x}}(t) - \underline{x}(t) \qquad (10)$$

From Appendix B, it is seen that the measured angles $\tilde{\theta}_1$ and $\tilde{\theta}_2$ which are the right ascension and declination of the landmark (sketch 1(b)) with respect to an inertial system with origin at the spacecraft, are not linearly related to the state vector but to first order, their deviations from their actual values are. That is,

$$\delta \hat{\theta} = H\, \delta \hat{\underline{x}} \qquad (11)$$

Considering only these first-order deviations, it can be stated that they are related after a measurement at time t_1 as the truly linear variables in equation A8.

That is,

$$\delta \hat{\underline{x}}'(t_1) = \delta \hat{\underline{x}}(t_1) + K(t_1)\left[\delta \tilde{\underline{\theta}}(t_1) - \delta \hat{\underline{\theta}}(t_1)\right] \qquad (12)$$

But,

$$\delta\hat{\underline{\theta}} = \hat{\underline{\theta}} - \underline{\theta} \tag{13}$$

since $\delta\hat{\underline{\theta}}$ is the deviation of the computed value $\hat{\underline{\theta}}$ from the true value $\underline{\theta}$. The deviation of the measured angle $\tilde{\underline{\theta}}$ from the true value is due to the noise in the measurement only,

$$\delta\tilde{\underline{\theta}} = \tilde{\underline{\theta}} - \underline{\theta} = \underline{\alpha} \tag{14}$$

So (12) becomes

$$\hat{\underline{x}}'(t_1) - \underline{x}(t_1) = \hat{\underline{x}}(t_1) - \underline{x}(t_1) + K\left[\underline{\alpha} - \hat{\underline{\theta}} + \underline{\theta}\right]$$

or

$$\hat{\underline{x}}'(t_1) = \hat{\underline{x}}(t_1) + K(t_1)\left[\underline{\theta} - \hat{\underline{\theta}} + \underline{\alpha}\right] \tag{15}$$

The matrix K is computed as in Appendix A.

THE MEASUREMENT ERRORS

The measurement error vector $\underline{\alpha}$ may be computed by considering the magnitude of a normally distributed error vector $\underline{d}$ perpendicular to $\underline{\rho}$ (appendix B) and also that this error vector is uniformly distributed between 0 and 2π about $\underline{\rho}$. This deviation may be expressed.

$$d = \rho\,\gamma$$
$$p\alpha_1 = d\cos\beta$$
$$\rho\,\alpha_2 = d\sin\beta$$

where γ is normally distributed with zero mean and β is uniformly distributed. The errors in the two angles are,

$$\alpha_1 = \frac{\rho}{p}\,\gamma\cos\beta = \frac{\gamma\cos\beta}{\cos\theta_2} \tag{16}$$

$$\alpha_2 = \gamma\sin\beta \tag{17}$$

The elements of R (Appendix A) may now be determined from the measurement errors, α_1, and α_2,

$$\mathcal{R} = \mathcal{E}\left[\underline{\alpha}\,\underline{\alpha}^T\right] = \begin{bmatrix} \overline{\alpha_1^2} & \overline{\alpha_1\alpha_2} \\ \overline{\alpha_2\alpha_1} & \overline{\alpha_2^2} \end{bmatrix}$$

The distribution functions of γ and β are,

$$f(\gamma) = \frac{1}{\sqrt{2\pi}\,\sigma_\gamma}\,e^{-\gamma^2/2\sigma_\gamma^2} \qquad -\infty < \gamma < \infty \tag{18}$$

$$f(\beta) = \frac{1}{2\pi}, \qquad 0 \leq \beta \leq 2\pi \tag{19}$$

Therefore,

$$\overline{\alpha_1^2} = \frac{\sigma_\gamma^2}{2\cos^2\theta_2} \qquad \text{and} \qquad \overline{\alpha_2^2} = \frac{\sigma_\gamma^2}{2}$$

also,

$$\overline{\alpha_1 \alpha_2} = \overline{\alpha_2 \alpha_1} = 0$$

which says that the errors are independent.

ORBITAL NAVIGATION SIMULATION PROGRAM (NAV)

Observing the form of equation (15) it is seen that the integrated values of the estimated state vector may be used in the state estimation procedure rather than the deviations from a reference or nominal trajectory. Equations (2) and (4) are integrated up to the measurement time. At the measurement time the state vector and the associated covariance matrix are updated by equations (15) and (A9).

The calculated angles $\hat{\theta}_1$, and $\hat{\theta}_2$ are computed with (B14) and (B15) from the estimated trajectory and the current best estimate of the landmark position. Equation (1) is integrated simultaneously with equation (2) to provide the actual trajectory, and assumed values of the actual landmark positions are input. These quantities are necessary in order to compute the actual angles θ_1 and θ_2 for use in the estimation equation, equation (15).

In the orbital navigation program the covariance matrix E is replaced by a matrix W, where

$$E = W W^T \tag{20}$$

This substitution is made to insure numerical stability. The covariance matrix may not remain positive definite after a large number of computations. The matrix W is guaranteed to remain at least positive semi-definite. The entire estimation problem may be formulated in terms of the W-matrix as given in Appendix C, and reference 2.

The covariance matrix is then computed in the program for information purposes from the W-matrix. The values of RMS position, velocity, and landmark position that may be given in this paper are computed from the trace of the covariance matrix,

$$
\begin{aligned}
\text{RMS position} &= \sqrt{E_{11} + E_{22} + E_{33}} \\
\text{RMS velocity} &= \sqrt{E_{44} + E_{55} + E_{66}} \\
\text{RMS landmark position} &= \sqrt{E_{77} + E_{88} + E_{99}}
\end{aligned}
\qquad (21)
$$

The gravity model used in the simulation consists of the moon as the primary, or reference body; the perturbations due to moon's triaxiality; the earth; the second, third, and fourth harmonics in the earth's potential function; and the sun. The trajectories are computed by the Encke integration technique.

SIMULATION RESULTS

A computer simulation was made to estimate the position and velocity of the Apollo spacecraft, at approximately 80 n.mi. altitude circular orbit above the moon, from sightings on five lunar landmarks on the front side with 3 sightings per landmark. Two runs were made for comparison. In the first run, figure 1, it was assumed that the landmarks were perfectly. In the second run, figure 2, it was assumed that the landmarks were known only to the accuracies given in Table 1. Both runs use a .003 radian uncertainty for the combination of sextant and navigation system errors.

At four hours from Lunar Orbit Insertion, which is approximately the time for beginning of Lunar Excursion Module descent, the RMS position and velocity for the case of perfect landmarks (figure 1) were 0.34 n.mi. and 1.4 fps. At the same time, for the case where the landmarks were poorly known, (figure 2) the RMS position and velocity were 0.9 n.mi. and 4.0 fps. Comparison of these results indicates the amount of error introduced if the landmarks are assumed to be known perfectly.

It is seen from figure 2, that the RMS position and velocity increase between measurements and decrease when a measurement is made, as would be expected. The net effect of the measurements is a reduction in RMS position and velocity. From figure 2c it is seen that the RMS landmark uncertainty also decreases, and the reduction is from just below 8000 feet to less than 3000 feet for the last three landmarks sighted.

CONCLUDING REMARKS

The analytic description of an orbital navigation program has been presented. This program will estimate the position and velocity of a spacecraft while in orbit about some central body using as navigational information the observed inertial positions of landmarks located on the central body. At the same time the program estimates the position of the landmarks with respect to a coordinate system fixed in the body.

This program has been applied to the problem of determining the orbit of the Command and Service Module between the times of Lunar Orbit Insertion and Lunar Excursion Module separation. It is shown that if the landmarks are assumed to be perfectly known, overoptimistic estimates of RMS position and velocity of the CSM at LEM separation will occur. Using realistic values of landmark uncertainties, the RMS position and velocity at LEM separation are about three times as high as for the same case using perfectly known landmarks. The landmark RMS positions are decreased by about a factor of two-thirds from their initial value.

REFERENCES

1. Positional Uncertainties in Lunar Landmarks, MSC Internal Note No. 65-ET-2, January 5, 1965.

2. Battin, R. H., Astronautical Guidance, McGraw-Hill, 1964.

3. Goss, R. D., and Muller, E.S., Deriving Random Error Vectors from Covariance Matrix, MIT/IL, SGA Memo #36, August 13, 1964.

4. Nance, R.L., Error Sources in Selenographic Positions Derived from Earth-Based Tracking of Spacecraft Orbiting the Moon, Proposed NASA TM.

APPENDIX A

THE OPTIMAL LINEAR FILTER

Given the linear dynamic system

$$\dot{\underline{x}} = F(t)\underline{x} \quad , \quad t \geq 0 \tag{A1}$$

Where $\mathcal{E}(\underline{x}(t_0)\,\underline{x}^T(t_0)) = \mu(0)$ is known and the

$\mathcal{E}((\hat{\underline{x}}(0) - \mu(0))(\hat{\underline{x}}(0) - \mu(0))^T) = E(t_0)$ is given

the system (A1) is observed through

$$\underline{\theta} = H(t)\underline{x} + \underline{\alpha} \tag{A2}$$

made at discrete times. α is white gaussian noise, with zero mean,

$$\mathcal{E}(\underline{\alpha}) = 0$$

also $\mathcal{E}(\underline{\alpha}(t)\underline{\alpha}^T(\tau)) = \delta(t - \tau)\mathcal{R}(t)$

and

$$\mathcal{R}(t) = \mathcal{R}^T(t) > 0$$

The best estimate of the initial conditions for (A1) are $\hat{\underline{x}}(t_0)$ which are given. The best estimate at a later time t_1 for x is,

$$\hat{\underline{x}}(t_1) = \Phi(t_1, t_0)\,\hat{\underline{x}}(t_0) \tag{A3}$$

where,

$$\dot{\Phi}(t,t_0) = F(t)\,\Phi(t,t_0), \qquad \Phi(t_0,t_0) = I$$

The error in the estimate is defined as,

$$\underline{e}(t) = \hat{\underline{x}}(t) - \underline{x}(t) \tag{A4}$$

The covariance matrix E is defined as,

$$E(t) = \mathcal{E}\left(\underline{e}(t)\,\underline{e}^T(t)\right) \qquad E(t) > 0 \tag{A5}$$

which is the solution to the Riccati equation,

$$\dot{E}(t) = F(t)E(t) + E(t)F^T(t) \tag{A6}$$

which is integrated with the initial conditions,

$$E(t_0) = \mathcal{E}\left[\underline{e}(t_0)\,\underline{e}^T(t_0)\right] \tag{A7}$$

If a measurement is made at t_1, the best estimate for $\underline{x}$ is assumed to be,

$$\hat{\underline{x}}'(t_1) = \hat{\underline{x}}(t_1) + K(t_1)\left[\tilde{\underline{\theta}}(t_1) - \hat{\underline{\theta}}(t_1)\right] \tag{A8}$$

where $\tilde{\underline{\theta}}(t_1)$ is the measurement at t_1 and $\hat{\underline{\theta}}(t_1) = H\hat{\underline{x}}(t_1)$ is what the measurement is expected to be.

The covariance matrix is updated at t_1 by

$$E'(t_1) = E(t_1) - K(t_1)H(t_1)E(t_1) \tag{A9}$$

The optimum value of the weighting matrix $K(t_1)$ is,

$$K(t_1) = E(t_1)H^T(t_1)M^{-1}(t_1) \tag{A10}$$

where, $$M(t) = H(t_1)E(t_1)H^T(t_1) + R(t_1) \tag{A11}$$

$K(t_1)$ is optimum in the sense that it minimizes the trace of $E(t_1)$, i.e., it minimizes the sum of the mean squared errors in the estimate of x. After the measurement, equations (A3) and (A6) are re-initialized and then integrated to the next measurement time t_2.

APPENDIX B
FORM OF THE H MATRIX

The H matrix for this problem is a two-by-nine augmented matrix, since there are two measurements and the state vector is a nine vector.

Specifically,

$$H = \begin{bmatrix} \underline{h}_1^T \\ \underline{h}_2^T \end{bmatrix} \tag{B1}$$

The vectors $\underline{h}_1$ and $\underline{h}_2$ are associated with the angles θ_1 and θ_2 respectively.

From sketch 1(b) it is seen that

$$\theta_i = \theta_i(\underline{\rho}) \qquad i = 1, 2 \tag{B2}$$

Taking the first variation,

$$\delta\theta_i = \left(\frac{\partial\theta_i}{\partial\underline{\rho}}\right)\delta\underline{\rho} \qquad i = 1, 2 \tag{B3}$$

But $\underline{\rho} = \underline{r}_L - \underline{r}$ and, $\underline{r}_L = A\underline{x}'_L$

so: $$\delta\underline{\rho} = A\,\delta\underline{x}'_L - \delta\underline{r} \tag{B4}$$

Substituting B4 into B3,

$$\delta\theta_i = -\left(\frac{\partial\theta_i}{\partial\underline{\rho}}\right)\delta\underline{r} + \left(\frac{\partial\theta_i}{\partial\underline{\rho}}\right)A\,\delta\underline{x}'_L \qquad i = 1, 2 \tag{B5}$$

It is desirable to obtain $\delta \underline{x}_L$ in selenographic latitude, μ , longitude, λ, and altitude, h , rather than in cartesian form.

$$x'_{L1} = (R_m + h)\cos\mu \cos\lambda$$
$$x'_{L2} = (R_m + h)\cos\mu \sin\lambda$$
$$x'_{L3} = (R_m + h)\sin\mu$$

Taking the variation,

$$\begin{pmatrix} \delta x'_{L1} \\ \delta x'_{L2} \\ \delta x'_{L3} \end{pmatrix} = \begin{bmatrix} -x'_{L3}\cos\lambda & -x'_{L2} & x'_{L1}/R_m{+}h \\ -x'_{L3}\sin\lambda & x'_{L1} & x'_{L2}/R_m{+}h \\ (R_m{+}h)\cos\mu & 0 & x'_{L3}/R_m{+}h \end{bmatrix} \begin{pmatrix} \delta\mu \\ \delta\lambda \\ \delta h \end{pmatrix}$$

or in vector-matrix notation,

$$\delta \underline{x}'_L = B\,\delta \underline{x}_L \tag{B6}$$

Substitute B6 into B5,

$$\delta\theta_i = -\left(\frac{\partial\theta_i}{\partial \rho}\right)\delta \underline{r} + \left(\frac{\partial\theta_i}{\partial \rho}\right) A B\,\delta \underline{x}_L \tag{B7}$$

Define,

$$\underline{b}_i^T = -\left(\frac{\partial\theta_i}{\partial \rho}\right) \tag{B8}$$

$$\underline{d}_i^T = \left(\frac{\partial\theta_i}{\partial \rho}\right) AB = -\underline{b}_i^T AB \tag{B9}$$

$$\underline{c}_i^T = 0 \tag{B10}$$

B7 becomes

$$\delta\theta_i = \underline{b}_i^T \delta \underline{r} + \underline{c}_i^T \delta \underline{v} + \underline{d}_i^T \delta \underline{x}_L$$
$$= \left[\underline{b}_i^T \;\; \underline{c}_i^T \;\; \underline{d}_i^T\right] \delta \underline{x} = \underline{h}_i^T \delta \underline{x}$$

since

$$\delta\underline{\theta} = \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix} \qquad \delta\underline{\theta} = \begin{pmatrix} \underline{h}_1^T \\ \underline{h}_2^T \end{pmatrix} \delta\underline{x} \tag{B11}$$

which is the same form as (A2) except for the noise. To first order, the expected deviation from the true value of the measurement vector is linearly related to the deviation in the state vector.

$$\delta\,\hat{\underline{\theta}} = H\,\delta\,\hat{\underline{x}} \tag{B12}$$

where

$$H = \begin{bmatrix} \underline{b}_1^T & \underline{c}_1^T & \underline{d}_1^T \\ \underline{b}_2^T & \underline{c}_2^T & \underline{d}_2^T \end{bmatrix} = \begin{bmatrix} \underline{b}_1^{*T} & \underline{d}_1^T \\ \underline{b}_2^{*T} & \underline{d}_2^T \end{bmatrix} \tag{B13}$$

The only vectors in H that are really necessary are $\underline{b}_1$ and $\underline{b}_2$ since $\underline{c}_1$ and $\underline{c}_2$ are zero and $\underline{d}_1$ and $\underline{d}_2$ are related by an orthogonal transformation to $\underline{b}_1$ and $\underline{b}_2$. From B8 it is seen that we need only the partial derivatives of θ_1 and θ_2 with respect to the components. These are readily obtainable from the geometry, since from sketch 1,

$$\cos\theta_1 = \frac{\rho_1}{P} \qquad P^2 = \rho_1^2 + \rho_2^2 \tag{B14}$$

$$\cos\theta_2 = \frac{P}{\rho} \qquad \rho^2 = P^2 + \rho_3^2 \tag{B15}$$

Therefore, it may be stated that,

$$\underline{b}_1^T = \left(\frac{\rho_2}{P^2} , \ \frac{-\rho_1}{P^2} , \ 0 \right) \tag{B16}$$

$$\underline{b}_2^T = \left(\frac{\rho_1 \rho_3}{P \rho^2} , \ \frac{\rho_2 \rho_3}{P \rho^2} , \ \frac{P}{\rho^2} \right) \tag{B17}$$

The orthogonality of the measurements θ_1 and θ_2 is immediately verified from B16 and B17 by noting that,

$$\underline{b}_1 \cdot \underline{b}_2 = 0 \tag{B18}$$

APPENDIX C

CONVERSION OF COVARIANCE MATRIX TO W-MATRIX FORMULATION

Because of numerical inaccuracies, that is, round-off and truncation errors, the covariance matrix E may not remain positive definite after many computational cycles. It is convenient to replace E by a matrix W such that,

$$E = WW^T \tag{C1}$$

The W matrix is guaranteed always to be at least positive semi-definite. The entire estimation problem may conveniently be formulated in terms of the W matrix.

The Riccatti equation (4) used to propagate the covariance matrix between measurements becomes,

$$\dot{W} = FW \tag{C2}$$

The covariance matrix is updated when a measurement is taken by

$$W' = WS^T \tag{C3}$$

where

$$S^TS = I - ZM^{-1}Z^T \tag{C4}$$

$$Z = W^TH^T \tag{C5}$$

$$M = Z^TZ + R \tag{C6}$$

The optimum weighting matrix K takes the form,

$$K = WZM^{-1} \tag{C7}$$

TABLE I

ERROR SOURCES FOR ONBOARD LUNAR ORBITAL NAVIGATION

(All errors are 1σ values)

1. Landmark Location Errors:

   (a) Uncertainty in moon-fixed (selenographic) system.*

ARC Dist. From (0°, 0°)	Horizontal Error	Vertical Error
0°	259 meters	980 meters
30°	670 meters	1000 meters
60°	1190 meters	820 meters

   (b) Landmark inertial orientation errors:**

   (1) Moon axial orientation
   (2) Moon rotation rate
   (3) Earth Moon vector

   500 meters each in landmark latitude, longitude, and altitude.

2. Acquisition Error by the Navigator:***

   1000 meters each in landmark latitude, longitude, altitude

3. Navigation System Errors:

   (a) Inaccuracy of sighting instrument:***
   (b) Misalinement of IMU
   (c) Lack of target resolution

   Total error = 3 m rad (1σ)

*Reference 1

**Reference 4

***Assumed realistic values.

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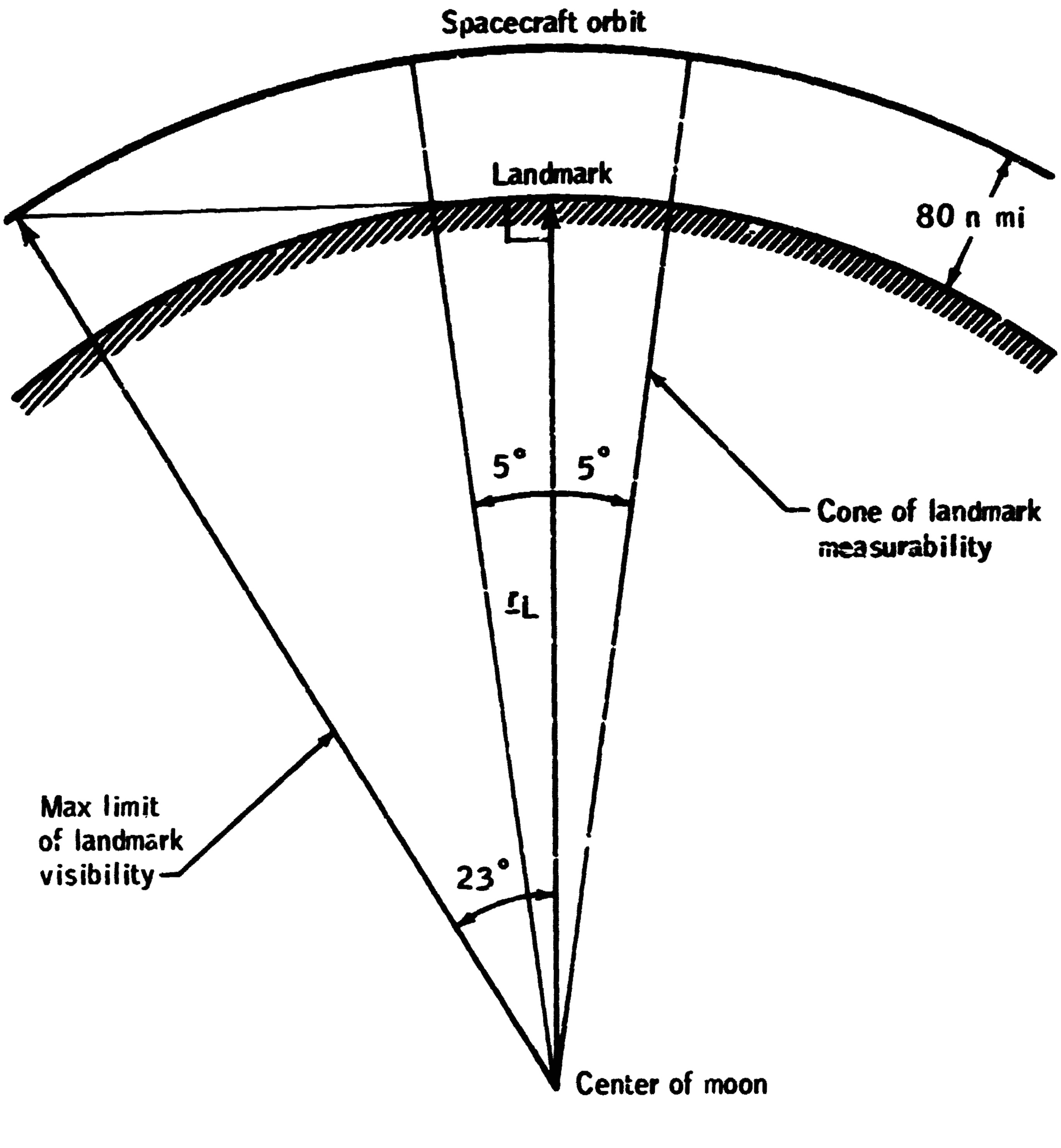

SKETCH 1a.- Observational limits.

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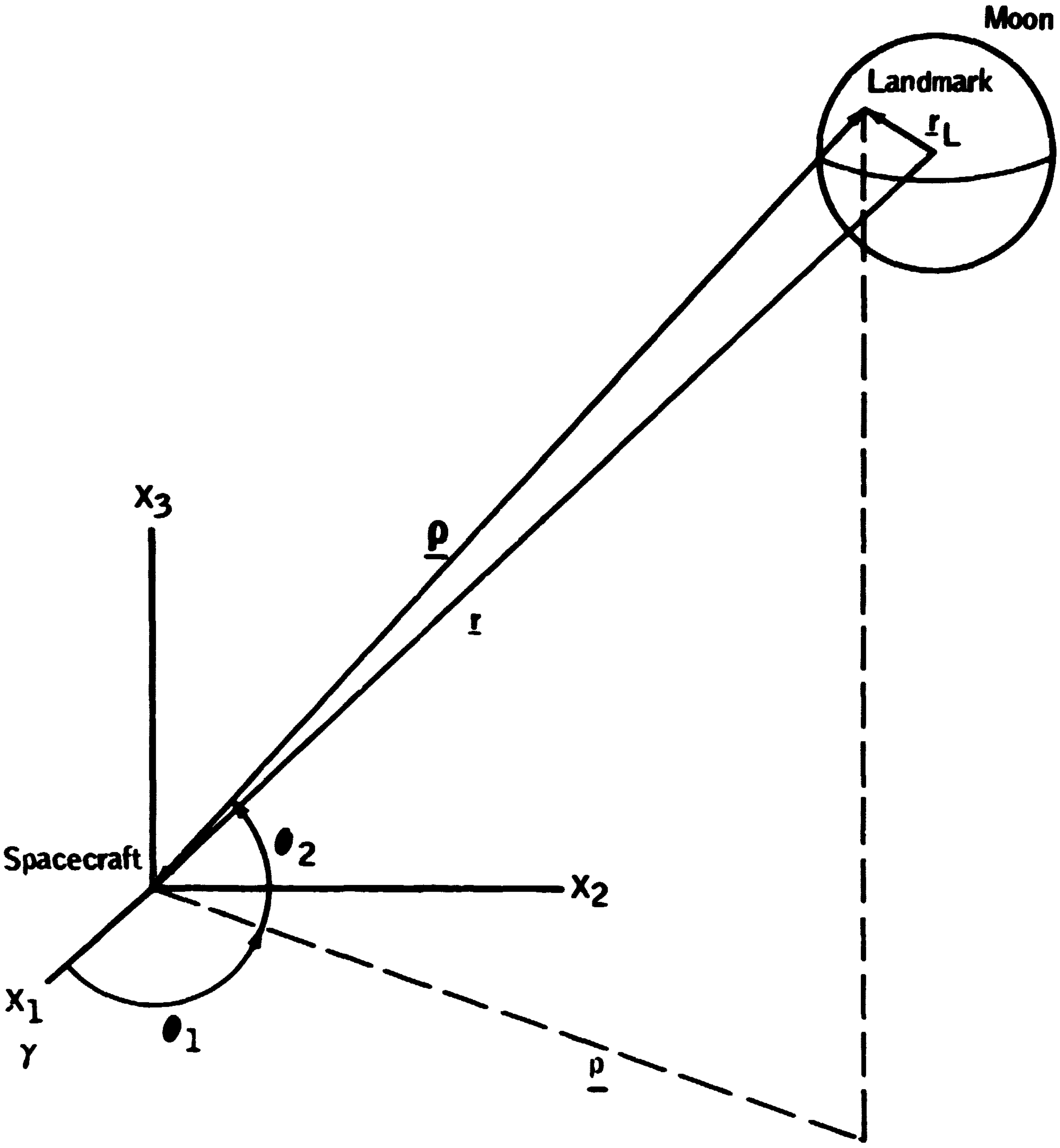
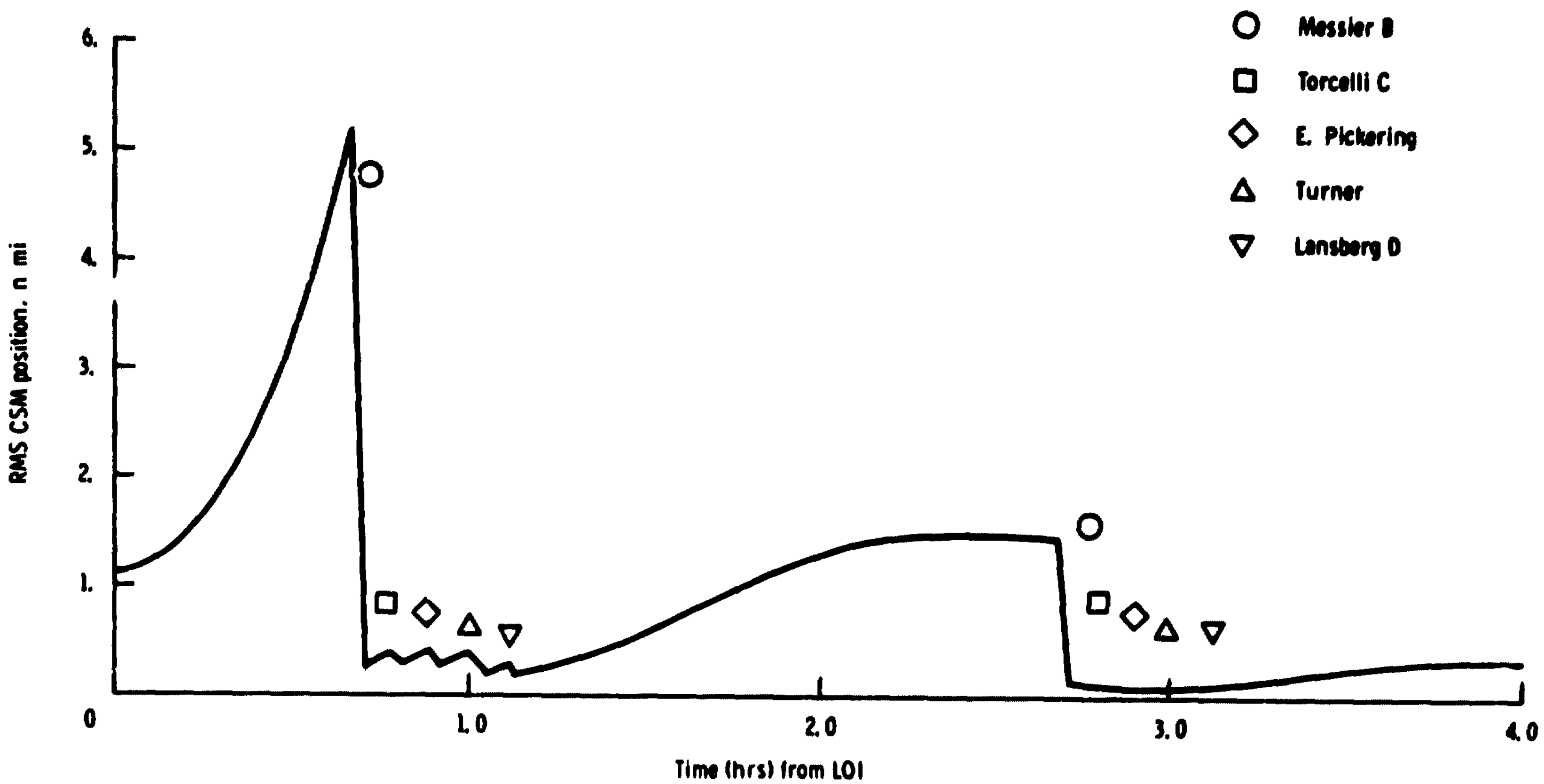


SKETCH 2a.- The measurement geometry.

NASA-S-66-584 JAN 20



3 sightings/landmark/orbit

Figure 1a. - RMS CSM position unc.'rtainty as a function of time from LOI - assuming no landmark errors.

NASA-S-66-586 JAN 20

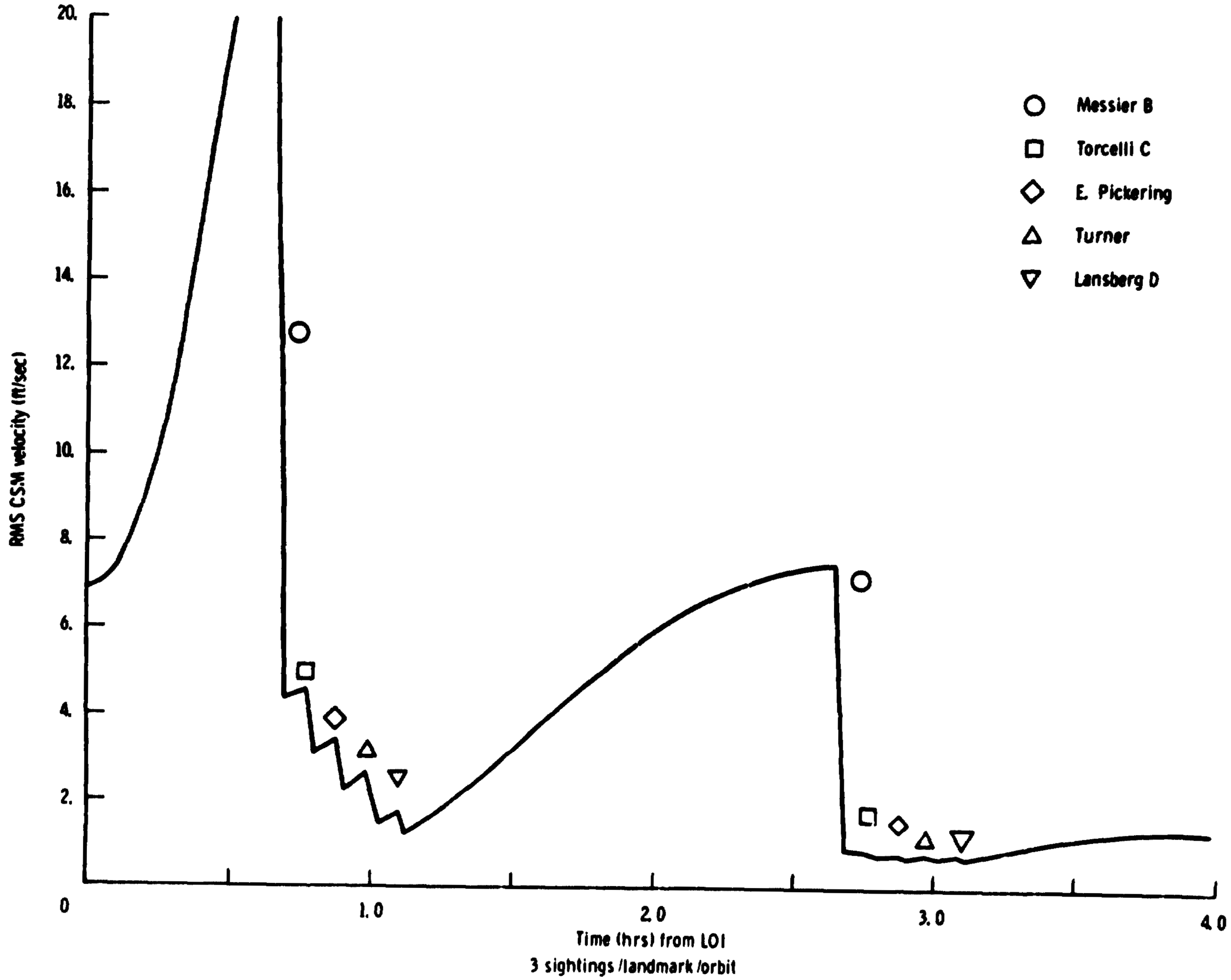
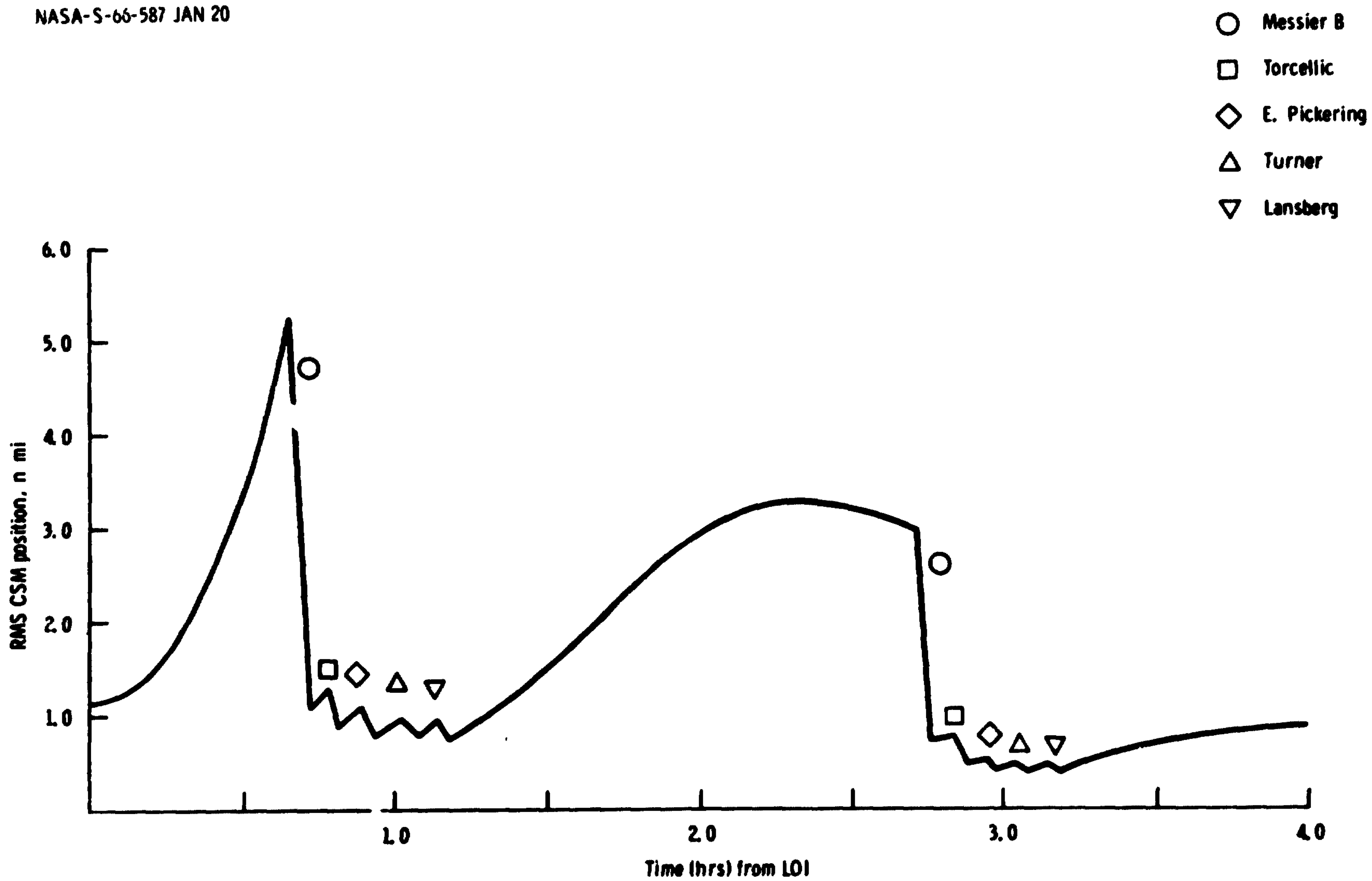


Figure 1b. - RMS CSM velocity uncertainty as a function of time from LOI. - assuming no landmark error.

NASA-S-66-587 JAN 20



3 sightings /landmark /orbit

Figure 2a. - RMS CSM position uncertainty as a function of time from LOI.

NASA-S-66-585 JAN 20

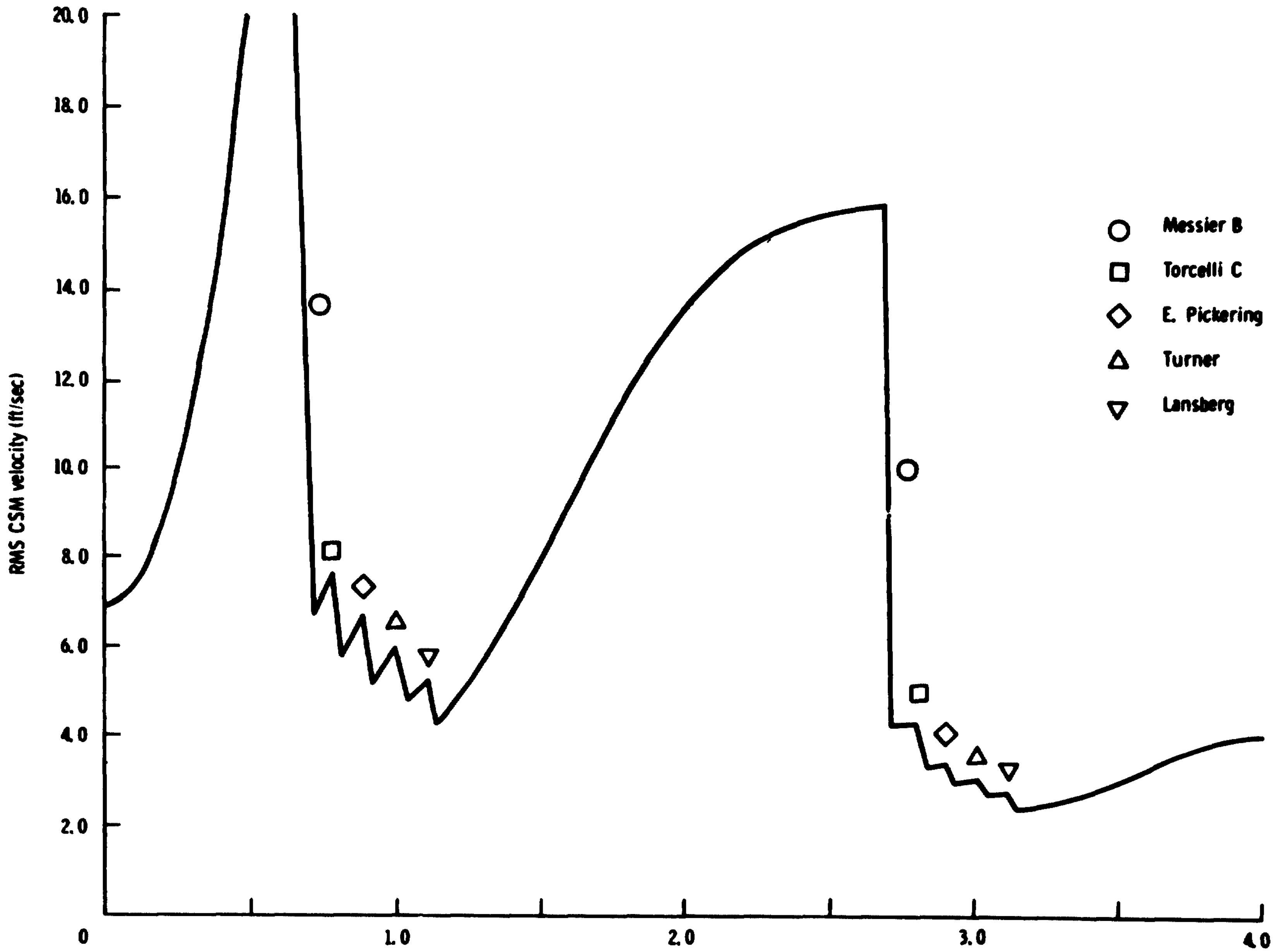


Figure 2b. - RMS CSM velocity uncertainty as a function of time from LOI

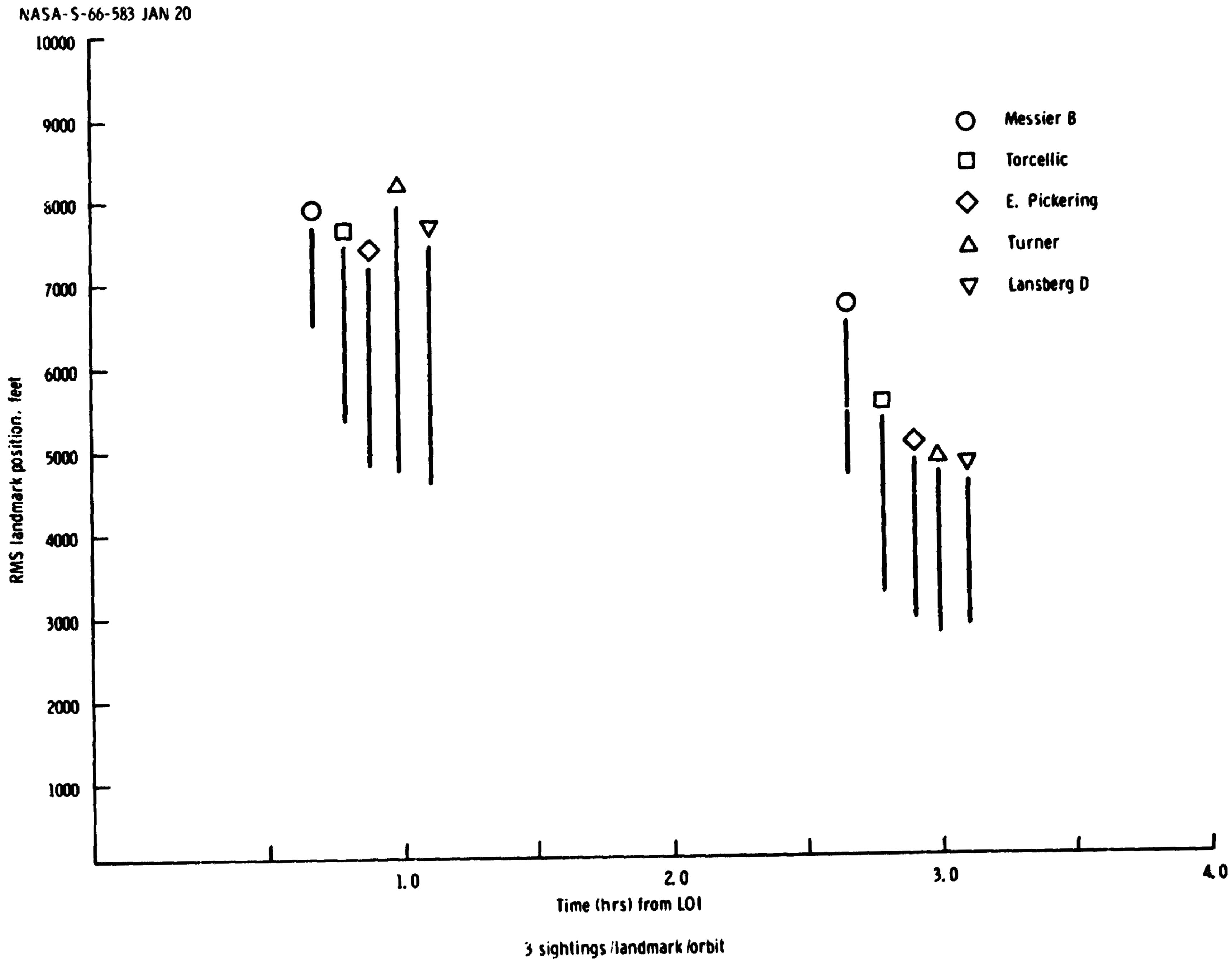


Figure 2c. - Reduction in landmark uncertainties as a function of time from LOI.